An efficient method for calculation of cooling in Lagrange computational gas dynamics

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Abstract

A new method for computation of gas cooling for Lagrange approach is suggested. The method is based on precalculation of cooling law for known cooling function. Unlike implicit methods, this method is very efficient, it is an one-step method which is even more accurate than implicit methods of the same order.

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1   Introduction

The Lagrange approach in gas dynamics assumes solution of the energy conservation equation in the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{P}{\rho}\nabla\mathbf{v} - \frac{\Lambda}{\rho}\,, \tag{1}$$

where u is the internal energy per unit mass, P is the pressure, $\mathbf{v}$ is the gas velocity, Λ is the cooling rate per unit volume that depends on gas density ρ and temperature T. To solve (1) the implicit methods are usually used, because the characteristic cooling time can be much smaller than gas-dynamical time. This circumstance makes the usage of explicit methods impossible.

An efficient one-step method for cooling computation for the energy conservation law in the form (1) is suggested in this paper. It is implied for the use in the computational gas dynamics and it is intended to predict variation of

the gas temperature over a time step. This method utilizes the information on given cooling function $\Lambda(\rho, T)$ and uses approximation in which cooling time is shorter than gas-dynamical time. The last condition is fulfilled in many gas-dynamical problems, particularly, in astrophysical problems which are in the area of the author's interest.

2 One-step method

Since cooling function depends on the temperature, let rewrite the equation (1) in the temperature and density terms. Assuming the equation of state of an ideal gas with mean molecular weight μ and adiabatic exponent γ, equation (1) gives

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -(\gamma - 1) \left[(\nabla \mathbf{v}) \, T + \frac{\mu}{k_{\mathrm{B}}} \frac{\Lambda}{\rho} \right] \, , \tag{2}$$

where k_{B} is the Boltzmann constant. The first term in the right-hand side of this equation conforms to purely adiabatic temperature change while the second one corresponds to cooling process. Following Sutherland & Dopita [1] and Nakasato et al. [2] we suppose cooling function to be proportional to the squared number density $\Lambda = n^2 \Lambda^*$, while Λ^* depends on the logarithmic temperature only:

$$\frac{\mathrm{d}\log T}{\mathrm{d}t} = -\frac{\gamma - 1}{\ln 10} \left[\nabla \mathbf{v} + \frac{\rho}{\mu k_{\mathrm{B}}} \frac{\Lambda^*(\log T)}{T} \right] \, , \tag{3}$$

where the definition $\rho = \mu n$ was used.

Despite of condition imposed on characteristic times, adiabatic cooling or heating may contribute significantly to temperature change. To account for this possibility and, simultaneously, to simplify the solving procedure, let to split the right-hand side of equation (3) into two independent processes, adiabatic and cooling ones. Since cooling time is short compared to gas-dynamical time, the velocity divergence and density may be taken as a constants. The resulting approximate expression for logarithmic temperature variation is

$$\log T = -\frac{\gamma - 1}{\ln 10} \, (\nabla \mathbf{v}) \, t + \sigma \, , \tag{4}$$

where σ satisfies the equation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\frac{\gamma - 1}{\ln 10} \frac{\rho}{\mu k_{\mathrm{B}}} \, 10^{-\sigma} \Lambda^*(\sigma) \tag{5}$$

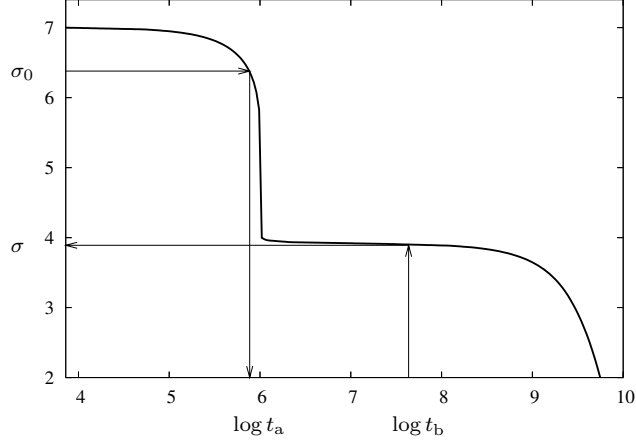with the initial condition $\sigma(t = 0) = \sigma_0 \equiv \log T_0$, where T_0 is the initial

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Fig. 1. The way to get the logarithmic temperature variation for given initial logarithmic temperature σ_0 and time interval $t_b - t_a$. The predicted value of σ used in (4) for temperature prediction.

temperature. Solution of this equation can be written implicitly in the form

$$\int_{\sigma_0}^{\sigma} \frac{\mathrm{d}x}{10^{-x}\Lambda^*(x)} = -\frac{\gamma - 1}{\ln 10} \frac{\rho}{\mu k_{\mathrm{B}}} t \; . \tag{6}$$

Equation (5) is an autonomous one. It means that the variation of σ depends on variation of t only. It's possible to use this property of equation (5) to precalculate the solution of (5). First, assume the cooling function Λ^* to be defined in the temperature interval $[T_{\min}, T_{\max}]$ or, in our case, in the corresponding interval $[\sigma_{\min}, \sigma_{\max}]$. Second, scale the time variable as

$$\tau = \frac{\gamma - 1}{\ln 10} \frac{\rho}{\mu k_{\mathrm{B}}} t \; . \tag{7}$$

And third, bin the values of integral (6) with lower bound fixed to $\sigma_{\min}$ and upper bound running from $\sigma_{\min}$ to $\sigma_{\max}$:

$$\int_{\sigma_{\min}}^{\sigma_i} \frac{\mathrm{d}x}{10^{-x}\Lambda^*(x)} = -\tau_i \; . \tag{8}$$

This binned function will represent the cooling process over entire range of temperatures where cooling function is defined.

An example of how to find σ for given initial logarithmic temperature σ_0 and time interval is shown in fig. (1). Here the cooling function of Sutherland & Dopita [1] (for $T > 10^4$ K) and Nakasato et al. [2] (for $T < 10^4$ K) was used (the function plotted corresponds to heavy elements abundance five orders of magnitude lower than the solar one). For initial logarithmic temperature σ_0 the corresponding initial time t_a in binned function (8) is searched first.

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Next, using the binned inverse function, the final value of σ is obtained for time moment, shifted to prediction time $t_{\rm b} = t_{\rm a} + t$. This algorithm needs two arrays, for logarithmic temperature and for scaled time. The actual values of both functions can be obtained by interpolation procedure defined for these arrays, i.e. just two array lookups are needed to get the temperature value by the expression (4).

It is interesting to estimate the error of the one-step method compared to the implicit method for solution of (1) or (2). Let denote $T_{\rm os}$ the solution derived by one-step method and $T_{\rm im}$ the solution derived by implicit method

$$\frac{T_{\rm im} - T_0}{t} = -(\gamma - 1) \left[(\nabla \mathbf{v}) \, T_{\rm im} + \frac{\rho}{\mu k_{\rm B}} \Lambda^* (\log T_{\rm im}) \right] . \tag{9}$$

It's easy to show that the errors of both methods arise in the second order. The second derivatives of deviation of approximate solution from the exact value are:

$$\epsilon_{\rm os} \equiv \left| \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\ln T_{\rm os} - \ln T \right) \right|_{T_0} = \frac{1}{\tau_{\rm gd} \, \tau_{\rm cool}} \left. \frac{\mathrm{d} \log (\Lambda^* / T)}{\mathrm{d} \log T} \right|_{T_0} , \tag{10}$$

$$\epsilon_{\rm im} \equiv \left| \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\ln T_{\rm im} - \ln T \right) \right|_{T_0}$$

$$= \left(\frac{1}{\tau_{\rm gd}} + \frac{1}{\tau_{\rm cool}} \right)^2 + \frac{1}{\tau_{\rm cool}^2} \left. \frac{\mathrm{d} \log (\Lambda^* / T)}{\mathrm{d} \log T} \right|_{T_0} + \epsilon_{\rm os} , \tag{11}$$

where the gas-dynamical time and cooling time are introduced:

$$\frac{1}{\tau_{\rm gd}} = (\gamma - 1) \, |\nabla \mathbf{v}| , \qquad \frac{1}{\tau_{\rm cool}} = (\gamma - 1) \frac{\rho}{\mu k_{\rm B}} \frac{\Lambda^* (\log T_0)}{T_0} . \tag{12}$$

It is clearly seen that the error of implicit method is systematically larger while the one-step method gives exact solution in pure adiabatic or pure cooling limit.

3 Conclusions

The presented method of calculation of cooling in Lagrange computational gas dynamics is faster and more precise than the implicit method of the same order.

4   Acknowledgements

References

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